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ON A NATURAL THEORY OF CONTINUOUS MEDIA

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Theoretical and experimental approaches are examined for describing mechanical and, in general, physical phenomena in continuous media in an accompanying reference frame using the local nonholonomic base frames of natural inertial reference frames. Such a natural description does not depend upon the properties and states of the "outside" observers. In the simulation of the physical space and of time by the spaces of Minkowski or Riemann in the general cases of accelerated motions of continua, accompanied by deformations for the moving system and for the system of observers, calculation algorithms are indicated for the tensor characteristics of the phenomena, described in the accompanying reference frame and in the systems of arbitrarily assigned observers.

For the scientific description of mechanical and, in general, physical objects, media, fields and phenomena it is necessary to use theoretical models. The mathematical modeling of a physical space and of time is the foundation of every theoretical interpretation of the world surrounding us. Until now the principal peculiarity of the usual representations of physical space and time has been the representation of a continuous four-dimensional continuum of points which can be given by four real numbers, namely, the coordinates, where one of these coordinates has a temporal nature, which finds its own reflection in the geometric properties of the four-dimensional continuum.

Let x^1, x^2, x^3, x^4 be the coordinates of the points and, by definition, let x^4 be the time coordinate. For fixed x^1, x^2, x^3 and variable x^4 we obtain a world line which in four-dimensional space can be treated as the trajectory of a point of a three-dimensional space, individualized by the values of coordinates x^1, x^2, x^3. Let us consider the functional relations

$$z^\alpha = \varphi^\alpha (x^\beta), \quad z^4 = f(x^\beta, x^4); \quad \alpha, \beta = 1, 2, 3 \tag{1}$$

In the coordinate system z^i $(i = 1, 2, 3, 4)$ we retain the definition of the individual points of a three-dimensional space and we retain the world lines; we change only the numerical values of the coordinates, namely, the names of the world lines, and we can arbitrarily change the origin of the reference frame and the scale of the time coordinate along the world lines. The coordinate systems x^i and z^i can be looked upon as the coordinates corresponding to one and the same family of world lines of individualized points forming an ideal entity-medium, being a three-dimensional collection of points taking different positions in a four-dimensional space as a function of the values of the time-coordinate x^4 or z^4. From now on the Latin indices vary from 1 to 4 and the Greek, from 1 to 3; a summation, i.e., a convolution with respect to like covariant and contravariant indices, is carried out everywhere in the formulas. It is evident that each isolated coordinate system x^i together with all possible transformations of form (1) define individual moving three-dimensional spaces which, depending on the definitions of the coordinates x^α and their ranges, are subspaces of the whole four-dimensional space or of some part of it. The system of individualized points in the different coordinate systems, subject to transformations (1) by definition, is called a reference frame.

The coordinates x^α or z^α are called the Lagrange coordinates of the corresponding reference frame. The systems of coordinates x^i and z^i are called the accompanying coordinate systems for the reference frame with Lagrange coordinates x^α or z^α. Thus, every coordinate system is an accompanying coordinate system for some reference frame. It is evident that for a given reference frame the indivual points in an accompanying coordinate system with coordinate lines modified as a function of the time coordinate are at rest since their three-dimensional coordinates x^α and z^α are constant.

In a given four-dimensional space we can examine numerous different reference frames. Let there be two different reference frames N and M with accompanying coordinate systems x^i and ξ^i corresponding to them and mutually related, in general, by one-to-one functional relations of the general form

$$x^i = f^i(\xi^1, \xi^2, \xi^3, \xi^4) \quad \text{or} \quad \xi^i = \varphi^i(x^1, x^2, x^3, x^4) \qquad (2)$$

In this case relations (2) define the law of motion of system M relative to N and, vice versa. By definition there occurs a motion of the system of individual points in M with $\xi^\alpha = \text{const}$ relative to the reference frame $N(x^i)$ or, conversely, of the system of individual points in reference frame N with $x^\alpha = \text{const}$ relative to the reference frame $M(\xi^i)$. For the sake of definiteness in what follows we agree to call the reference frame N with coordinates x^i the observer system and the reference frame M with accompanying coordinates ξ^i, the moving system. The systems of world lines and coordinate lines both in system N as well as in system M can be of the most general form.

To describe the geometric properties of the space and the typical properties of motion (2) and of the other physical phenomena we introduce the typical concepts of scalar, vector and, in general, tensor nature, and, in particular we introduce by well-known means the covariant ∂_i and contravariant ∂^i basic vectors $i = 1, 2, 3, 4$, for every coordinate system. To compare typical vector and tensor quantities at different points of space-time we introduce, by means of a far reaching specialization of the properties of the space, the connection coefficients $\Gamma_{ij\cdot}^{\ \ k}(x^1, x^2, x^3, x^4)$ occurring in the formulas

$$\partial \partial_i / \partial x^j = \Gamma_{ij\cdot}^{\ \ k} \partial_k \qquad (3)$$

The subsequent paths of physico-geometric concretization of the mathematical model of space and time can be different. As is well known, as the model of a physical space in Newtonian mechanics we introduce a three-dimensional Euclidean space, while time is treated as a scalar. In the special theory of relativity (STR) it is postulated that a pseudo-Euclidean space serves as the model of the four-dimensional physical space-time, while in the general theory of relativity (GTR) and in numerous generalizations, it is accepted that the four-dimensional space-time is Riemannian and that locally this space is pseudo-Euclidean just as in the STR. Many other suggestions for interpreting the models of physical space are also known.

Below we consider only kinematic problems and accept that space-time forms a Riemannian or a pseudo-Euclidean space in which the following relations hold. The metric has the form

$$ds^2 = g_{ij}(x^k)\,dx^i dx^j \tag{4}$$

and

$$\Gamma^k_{ij\cdot} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

where ds is the line element of the elementary vector $d\mathbf{s} = dx^i \mathbf{\partial}_i$, and g_{ij} are the components of the metric tensor. By using a transformation of form (1) with function f defined to within an additive constant, at any point of space $N\,(z_0{}^l)$ the expression for ds^2 can be brought to the form

$$ds^2 = (dz^4)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 \tag{5}$$

Both in the GTR as well as in the STR such a transformation of form (1) is possible only locally at a point of $N\,(z_0{}^l)$. By any Lorentz transformation of form (2), being a linear transformation from z^i to y^i and containing ten arbitrary constants in the general case, formula (5) is reduced to the form

$$ds^2 = (dy^4)^2 - (dy^1)^2 - (dy^2)^2 - (dy^3)^2, \quad \Gamma_{ij\cdot}{}^k\,(y_0{}^l) = 0 \tag{6}$$

Using a general transformation of form (2), the transformation to form (6) in the STR is possible globally at all points of the space at once. As is well known [1], in the GTR, in the general case, equality (6) can be realized at once at all points of any curve C by means of a coordinate transformation of general form (2). The corresponding coordinates are determined to within any Lorentz transformation; they are connected with the form of curve C and are called the Fermi coordinates. In this case curve C at all its points can be an accompanying world line in the coordinate system y^i only if this curve C is a geodesic.

If the components of the metric tensor are like in different coordinate systems x^i and $x'^i\,(x^k)$, i.e.,

$$g_{ij}'\,(x'^k) = g_{ij}\,(x'^k) \tag{7}$$

then from the point of view of the metric properties of Riemannian space there is a symmetry relative to those various coordinate system transformations which can form a finite or infinite group of coordinate transformations. In the general case Riemannian spaces are asymmetric. For asymmetric Riemannian spaces the values of the metric tensor components, taken at all points of the space, completely determine a coordinate system which, obviously, has an invariant geometric sense [2]. From what has been said it follows that for asymmetric Riemannian spaces transformation (2) or the law of motion of the accompanying reference frame M with coordinates ξ^i relative to the observer system N with coordinates x^i can be found from the equations

$$g_{pq}'(\xi^l) = g_{ij}(x^k)\,\frac{\partial x^i}{\partial \xi^p}\,\frac{\partial x^j}{\partial \xi^q} \tag{8}$$

if $g_{ij}\,(x^k)$ and $g_{pq}'\,(\xi^l)$ are known (for instance, from experiment). It is obvious that for the solvability of the problem of integrating system (8), which consists of ten partial differential equations with four unknown functions $x^i\,(\xi^k)$, the functions $g_{ij}\,(x^k)$ and $g_{pq}'\,(\xi^l)$ must satisfy appropriate compatibility conditions.

The compatibility of Eqs. (8) can in principle serve as a check on the fitness of modeling space and time by a metric of form (4).

On the other hand, the law of motion (2) of the continuous medium and the metric tensor components $g_{pq}'(\xi^l)$ and $g_{ij}(x^k)$ can be determined theoretically by using dynamic equations or by using measurements in system x^k of the signals carrying information on events in the moving system ξ^k. The latter way is not always possible in principle and is connected with a number of distortions brought in, firstly, by the singularities and processes in the intervening medium in which the signals are propagated and, secondly, by the difference in the mechanical and, in general, physical state of the observer and the moving medium. Examples are, on the one hand, the phenomena of scattering, absorption and divergence of electromagnetic or acoustic perturbations in the intervening medium and every kind of "noise", and on the other hand, phenomena similar to the Doppler effect and especially to relativistic effects, connected with the difference in electromagnetic characteristics and with the difference in the course of time both in the change of geometric sizes from the viewpoint of the observer in the observer system as well as in the moving system.

It is obvious that the examples of the effects listed are connected in an essential way with the choice of the observer system and reflect not only the properties and states of the phenomena in the moving system, but also both the phenomena and the states of the observer himself, which are, in general, immaterial from the point of view of the physical laws regulating the phenomena taking place in the moving system. In this regard we can take the general position that it is advisable to investigate and to establish the fundamental physical relations by constructing and measuring theoretical models directly in the accompanying reference frame by theoretical or experimental means, and after this to recompute and reformulate the results obtained in the point of view of the interested observer. Essentially, in the simplest special cases, such things enter into the physical theories sometimes not entirely in explicit form. Later we show that in their essence the results obtained in the accompanying reference frame, independent of the state and motion of the random observers, have a simpler form than the results obtained after the above-mentioned reformulation which can be carried out by using additional rather complex mathematical operations constituting navigational calculations. Roughly speaking, the crux of the matter can be illustrated by examples of comparing the viewpoint on Ptolemy in an Earth observer system with the viewpoint of Copernicus in a system accompanying the center of gravity of the solar system, or by example of considering the motion of Jupiter's satellites relative to its center, or by considering the physical phemomena in the atoms of a moving body when using reference frames attached to the atom nuclei. etc.

Below we dwell on methods for the experimental determination of the components of the metric tensor in the accompanying coordinate system and on mathematical navigation problems of the recalculation of given tests in the accompanying reference frame on the measurement results of the metric tensor components obtained in the tests or in the theory by a specified observer.

At each point M on some world line C with an appropriate coordinate system z^i, at which relations (5) are fulfilled, we determine the coordinate frames ϑ_i' of unit vectors. If we accept that the spatial frames $\vartheta_1', \vartheta_2', \vartheta_3'$, perpendicular to the four-dimensional velocity $\bar{u} = \vartheta_4'$, do not rotate in the three-dimensional space with coordinates z^α when passing from one point to an adjacent one along C, then such frames form a moving vector frame along C, called the Fermi-Walker frame. From a consideration of an infinitesimal Lorentz transformation of the Fermi—Walker vector frame when passing from point M to a point infinitely close, it is easy to deduce that along the world line there holds at each point M of it the vector equalities

$$d\vartheta_i' / ds = (a_l u_i - a_i u_l) \vartheta'^l \qquad (\mathbf{a} = d\mathbf{u}/ds = a^\alpha \vartheta_\alpha') \tag{9}$$

where $\mathbf{a}$ is the four-dimensional acceleration of point M.

Together with frame ϑ_i' we consider a frame ϑ_i^* for which the equalities $\vartheta_i^* = \vartheta_i'$, but $d\vartheta_i^* / ds = 0$ or

$$\mathbf{a}_1 = d\vartheta_4^* / ds = d\mathbf{u}^* / ds = 0$$

are true at each point M. Frame ϑ_i^* can be treated as the Fermi—Walker frame for the geodesic line passing through point M of the given world line and tangent to it at this point. The vector basis ϑ_i^* at point M corresponds to a local inertial reference frame. In this frame the point M^* coinciding with the given point M on the world line being examined moves along the geodesic without acceleration, $\mathbf{a}_1^* = 0$, but its four-dimensional velocity at the instant being considered exactly equals the velocity of point M ($\mathbf{u}^* = u$). (It is evident that the three-dimensional velocities of points M and M^* relative to any reference frame are alike as well). The local basis ϑ_i^* can be treated at each point of the accompanying reference frame as an attribute of this reference frame and as a local inertial basis defining the progressively freely falling vector frame ϑ_i^* in a gravitational field. The locally defined inertial reference frame ϑ_i^* is called the natural frame for the point M (ξ^α, s) being examined on the world line in the accompanying system.

It is easy to understand that the three-dimensional acceleration vector $\mathbf{a}^*$ of the points of the material medium, measured by a very small three-component accelerometer fixedly attached to basis ϑ_1', ϑ_2', ϑ_3', equals the acceleration of the sensing element (for example the acceleration of a small bead of mass m) of the accelerometer relative to the inertial basis ϑ_i^*. This acceleration equals zero when point M moves along a geodesic (a free fall in a gravitational field, corresponding to the weightless state). Here we accept that

$$\frac{d^2\mathbf{r}^*}{ds^2} = \frac{1}{c^2}\frac{d^2\mathbf{r}^*}{d\tau^2} = \frac{1}{c^2}\mathbf{a}^*$$

where $d\mathbf{r}^*$ is an elementary shift of point M relative to basis ϑ_i^*, c is the "velocity of light", and $d\tau$ is an increment in proper time, alike in bases ϑ_i' and ϑ_i^* and coinciding with the increment of proper time along the world line being examined, passing through the point M being considered ($d\mathbf{r}^* / d\tau = 0$ but, in general, $d^2\mathbf{r}^* / d\tau^2$ is nonzero). The conclusion arrived at above follows immediately from the equality (see [3])

$$\mathbf{a} = \frac{d\mathbf{u}}{ds} = \frac{1}{c^2}\mathbf{a}^* \tag{10}$$

In the accompanying reference frame, the frame supporting the accelerometers and corresponding to the bases $\vartheta_\alpha^* = \vartheta_\alpha'$ ($\alpha = 1, 2, 3$), keeping its spatial orientation fixed, can be realized by means of three gyroscopes moving freely in a gimbal suspension with noncoplanar axes.

We now describe a possible method for determining the metric tensor components $g_{\hat{pq}}(\xi^k)$ $(p,\ q = 1, 2, 3, 4)$ in the accompanying reference frame for a material medium. It is obvious that without loss of generality we can introduce new accompanying coordinates for any reference frame by using transformations of form (1), in which the following equality is fulfilled: $g_{\hat{44}} = c^2$, where c is a constant which, according to the fundamental postulate of the STR and the GTR, can be taken as equal to the velocity of light; therefore, in the case of an arbitrary accompanying reference frame the formula for ds^2 can be taken in the form

$$ds^2 = c^2 d\tau^2 + 2g_{\hat{\alpha 4}} d\xi^\alpha d\tau + g_{\hat{\alpha\beta}} d\xi^\alpha d\xi^\beta \tag{11}$$

We see that on each world line with $\xi^\alpha = \text{const}$, since $c d\tau = ds$, we can treat the coordinate τ as an invariantly defined time, measured on small clocks firmly attached to the fixed points in the accompanying reference frame. The origin of the proper time reference frame on the different world lines can be established independently and, in general, arbitrarily.

It is evident that the system of world lines in the general case does not admit of families of orthogonal three-dimensional surfaces. Such surfaces could be introduced as the surfaces

$$\tau(\xi^1,\ \xi^2,\ \xi^3) = \text{const}$$

Therefore, in the general case it is not possible to make all the $g_{\hat{\alpha 4}}(\xi^\beta, \tau)$ zero at once by using a holonomic transformation of form (1), since otherwise when $g_{\hat{\alpha 4}} = 0$ the coordinate surfaces $\tau = \text{const}$ would be orthogonal to the world lines; as a consequence of this in the general case when $g_{\hat{\alpha 4}}(\xi^\beta,\ \tau) \neq 0$ it is not possible, by choosing the origin of the proper time reference frame on the world lines, to find on all the world lines points corresponding to $\tau = \text{const}$, i.e., points corresponding to one and the same instants of proper time τ. Hence it follows that for a moving medium it is impossible in the general case to find a finite or infinite body forming the whole medium or a part of it and occupying infinite or finite three-dimensional volume, for which we could find a proper time which is one and the same for all points of such a body.

At each point of the medium we can rewrite the expression for ds^2 in (11) as

$$ds^2 = c^2 d\tau_1^2 + h_{\hat{\alpha\beta}} d\xi^\alpha d\xi^\beta \tag{12}$$

$$d\tau_1 = d\tau + \frac{g_{\hat{\alpha 4}}}{c^2} d\xi^\alpha, \quad h_{\hat{\alpha\beta}} = g_{\hat{\alpha\beta}} - \frac{g_{\hat{\alpha 4}} g_{\hat{\beta 4}}}{c^2} = (\overline{\partial}_\alpha \overline{\partial}_\beta) \tag{13}$$

$$\overline{\partial}_\alpha = \partial_{\hat{\alpha}} - \frac{g_{\hat{\alpha 4}}}{c^2} \partial_{\hat{4}}, \quad \partial_{\hat{i}} = \frac{\partial \mathbf{r}}{\partial \xi^i}$$

where $\partial\mathbf{r}$ is an element of the coordinate line with number i. The vectors $\partial_{\hat{i}}$ form a basis in the accompanying coordinate system corresponding to formula (11). The vectors $\overline{\partial}_\alpha$ form the accompanying basis for the small three-dimensional volume element dv_3 of the space orthogonal at point M to the world line being examined. As the accompanying coordinates of the individual points in the infinitesimal volume dv_3 we can take $d\xi^\alpha$, where $\partial_\alpha = \partial_1 \mathbf{r} / \partial \xi^\alpha$ ($\partial_1 \mathbf{r}$ is an element of the corresponding coordinate line in volume dv_3). Along each world line we have $\xi^\alpha = \text{const}$ and, therefore, $d\tau_1 = d\tau$ on C. The quantity τ_1 defined from (13), in contrast to variable τ, is defined nonholonomically in a four-dimensional volume, since the right-hand side in the expression for $d\tau_1$ is not a total differential because in the general case the integrability conditions

$$\frac{\partial g_{\hat{\alpha 4}}}{\partial \tau} = 0, \quad \frac{\partial g_{\hat{\alpha 4}}}{\partial \xi^\beta} - \frac{\partial g_{\hat{\beta 4}}}{\partial \xi^\alpha} = 0 \tag{14}$$

are not satisfied. We can take ds^2 globally to form (12) with the aid of holonomic transformation (1) only when the integrability conditions (14) are satisfied, and, thus, make all the $g_{\hat{\alpha}}$, zero, draw the orthogonal surfaces to the given family of world lines and set $\tau_1 = \tau$ in the finite volumes. The corresponding reference frame defined by the world line family being examined and the coordinate system are said to be synchronous. In a synchronous reference frame the proper time τ can be introduced as a global characteristic for the corresponding three-dimensional medium. The three-dimensional metric

$$dl^2 = -h_{\hat{\alpha\beta}} d\xi^\alpha d\xi^\beta$$

holds in synchronous systems on the three-dimensional surfaces $\tau = \tau_1 = \text{const}$ orthogonal to the world lines. It is easy to see immediately from the geodesic equations that for a holonomically defined variable τ_1 and, consequently, in the presence of equalities (14), the world lines $\xi^\alpha = \text{const}$ coincide with the coordinate lines τ_1 and are geodesics. The motion of the medium's points on the geodesic world lines is inertial since the equality $\mathbf{a} = c^{-2}\mathbf{a}^* = 0$ is true along geodesics.

In every Riemannian space we can introduce synchronous reference frames, and, respectively, coordinate systems accompanying the world line family formed by the geodesics; however, in the general case the reference frame and the coordinate system connected with the family of geodesics as world lines are not synchronous. In Riemannian spaces synchronous systems are analogous to inertial frames in the STR and in Newtonian mechanics. In the general case of an asymmetric Riemannian space we can construct a synchronous coordinate system in a natural way with the aid of of the following construction. In the region being considered of the four-dimensional space we can uniquely pick out geometrically a certain point P and a certain direction at this point (for example. a point corresponding to the singular values of the invariants of the curvature tensor). Through point P we draw a geodesic L° in the time-like direction picked out and through point P we draw all possible geodesics orthogonal to L° at point P. The family of geodesics obtained forms a specific three-dimensional space Σ immersed in the given four-dimensional Riemannian space. Through each point of Σ we draw a geodesics L orthogonal to Σ. We treat the family of geodesics L, containing L° as well, as world lines corresponding to the reference frame. In this reference frame we introduce the accompanying coordinates x^1, x^2, x^3, τ, where τ is the proper time along geodesics L and the x^α are coordinates of the points of Σ. From the construction of the world lines it follows that because the lines L are orthogonal to Σ the formula (11) for ds^2 has the form

$$ds^2 = c^2 d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta \tag{15}$$

at the points of the three-dimensional space Σ, i.e., $g_{\alpha 4} = 0$ on Σ, and the equation of the three-dimensional surface Σ can take the form

$$\tau(x^1 x^2 x^3) = \tau_0$$

and we have $ds = cd\tau$ along the geodesics corresponding to $x^\alpha = \text{const}$, $\alpha = 1, 2, 3$. It is easy to perceive that when $\tau = \tau' \gtrless \tau_0$ the metric (15) retains its form, i.e., when $\tau \gtrless \tau_0$ the equalities $g_{\alpha 4}(x^\beta, \tau) = 0$ are true as well. As a matter of fact, since the coordinate lines τ are geodesics when $x^\alpha = \text{const}$, it follows from the equations of the geodesics

$$\frac{d^2x^i}{ds^2} + \frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma^i_{jk} = 0$$

that

$$\Gamma^i_{44} = \frac{1}{2} g^{ik}\left(2\frac{\partial g_{k4}}{\partial\tau} - \frac{\partial g_{44}}{\partial x^k}\right) = g^{i\alpha}\frac{\partial g_{\alpha 4}}{\partial \tau} = 0 \quad \text{or} \quad \frac{\partial g_{\alpha 4}}{\partial\tau} = 0$$

therefore $g_{\alpha 4}(x^1, x^2, x^3, \tau) = 0$ since $g_{\alpha 4}(x^1, x^2, x^3, \tau_0) = 0$ by construction.

Using an analogous construction we can in a unique manner reduce the three-dimensional form for dl^2 on the asymmetric three-dimensional surface Σ to

$$dl^2 = (dx^1)^2 + g_{\check{2}\check{2}}(dx^2)^2 + g_{\check{3}\check{3}}(dx^3)^2 + 2g_{\check{2}\check{3}}\, dx^2 dx^3$$

when $\tau = \tau_0$ and we can take $x^\alpha = 0$ at point P ; to the tangents to the orthogonal coordinate geodesics x^1 and x^2 on Σ at point P we can attach invariantly specified directions and, in addition, satisfy the following equalities:

$$g_{\check{2}\check{3}}(0, x^2, x^3, \tau_0) = 0, \qquad g_{\check{2}\check{2}}(0, x^2, x^3, \tau_0) = g_{\check{3}\check{3}}(0, 0, 0, \tau_0) = 1$$

In the case of symmetric Riemannian spaces the transformation of ds^2 to the form indicated or to a still simpler form also is possible, but the corresponding transformation is not unique.

To determine the components of the three-dimensional metric tensor $h_{\hat{\alpha}\hat{\beta}}(\xi^\alpha, \tau)$ at point M it is sufficient to establish in the three-dimensional infinitesimal element dv_3 of the space, perpendicular to vector **u** and directed along the tangent to the given world line, a linear transformation between the vectors of bases $\bar{\partial}_\beta$ and ∂_γ^* $\beta, \gamma = 1, 2, 3)$

$$\bar{\partial}_\beta = l_{\beta\cdot}{}^{\cdot\gamma}(\tau)\partial_\gamma^* \quad \text{or} \quad \partial_\gamma^* = c_{\gamma\cdot}{}^{\cdot\beta}(\tau)\,\bar{\partial}_\beta \; (l_{\alpha\cdot}{}^{\cdot\gamma} c_\gamma{}^{\cdot\beta} = \delta_\alpha{}^\beta) \tag{16}$$

It is necessary to construct a dynamic theory for the theoretical determination of matrix $l_{\beta\cdot}{}^{\cdot\gamma}$. For the experimental determination of matrix $l_{\beta\cdot}{}^{\cdot\gamma}$ it is sufficient to measure, using scaling devices in the proper reference frame, the six independent angles from the ten between the vectors $\bar{\partial}_\beta$ and the three unit vectors ∂_α^*, forming in a known way an unchangeably directed frame connected with the gyroscopes, and, in addition, it is sufficient to measure the three lengths of the vectors $\bar{\partial}_\alpha$. If the accompanying frame is absolutely rigid, then it is sufficient to measure only the three angles defining the orientation of the unalterable trihedron $\bar{\partial}_\beta$ in relation to the fixedly oriented trihedron ∂_α^*. After the determination of the matrix $l_{\beta\cdot}{}^{\alpha}$, from (13) we find

$$h_{\hat{\alpha}\hat{\beta}} = g_{\hat{\alpha}\hat{\beta}} - \frac{g_{\hat{\alpha}4}g_{\hat{\beta}4}}{c^2} = l_{\alpha\cdot}^{\cdot\gamma} l_{\beta\cdot}^{\cdot\mu} g^*_{\gamma\mu}, \quad g^*_{\gamma\mu} = (\partial_\gamma^*, \partial_\mu^*) \tag{17}$$

Without loss of generality we can take it that

$$g_{\gamma\gamma}^* = 1, \quad g_{\gamma\mu}^* = 0 \quad \text{for} \quad \gamma \neq \mu$$

in other words, we can reckon that the unit vector basis ∂_α^* is orthogonal.

Let us now give the formulas for determining $g_{\hat{\alpha}4}$ in the terms of the com -

ponents $a^{*\alpha}$ of the three-dimensional acceleration measured by an accelerometer and equal to the kinematically invariantly determined four-dimensional acceleration multiplied by c^2 for the point M on the world line C. In the different bases equality (10) can be further rewritten as

$$\left(\frac{d^2x^i}{ds^2}+\frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma^{\circ i}_{jk}\right)\partial_i^{\circ}=\frac{d^2y^i}{ds^2}\partial_i=\frac{d^2z^\alpha}{ds^2}\partial_\alpha^*= \tag{18}$$

$$\frac{1}{c^2}a^{*\alpha}\partial_\alpha^*=\left(\frac{d\tau}{ds}\right)^2\Gamma^{\hat{}i}_{44}\partial_i^{\hat{}}$$

In the accompanying system (11), $d^2\xi^i / ds^2 = 0$ and $d\tau / ds = 1 / c$; here ∂_i° are the basis vectors in system x^i; ∂_i are the basis vectors in system y^i which can be treated as a global cartesian coordinate system in the STR or as the holonomic Fermi coordinate system in the GTR for the given world line C. Since

$$\Gamma^{\hat{}i}_{44}=g^{\hat{}is}\frac{\partial g^{\hat{}}_{s4}}{\partial\tau}$$

the last of equalities (18) yields

$$\frac{1}{c^2}g^{\hat{}is}\frac{\partial g^{\hat{}}_{s4}}{\partial\tau}\partial_i^{\hat{}}=\frac{a^{*\alpha}}{c^2}\partial_\alpha^*$$

and after a scalar multiplication of the right and left hand sides by $\partial_j^{\hat{}}$ we obtain

$$\frac{\partial g_{\hat{i}4}}{\partial\tau}=a^{*\alpha}(\partial_\alpha^*\partial_j^{\hat{}})$$

Since ∂_α^* is perpendicular to $\partial_4^{\hat{}}$ and $g^{\hat{}}_{44}=c^2$, we have that $\partial g^{\hat{}}_{44}/\partial\tau=0$ and $(\partial_\alpha^*\ \partial_4^{\hat{}})=0$ are true when $j=4$. On the basis of formulas (16) and equality

$$\overline{\partial}_\alpha=\partial_\alpha^{\hat{}}-\frac{g^{\hat{}}_{\alpha4}}{c^2}\partial_4^{\hat{}}$$

we obtain

$$\frac{\partial g^{\hat{}}_{\beta4}}{\partial\tau}=a^{*\alpha}(\partial_\alpha^*\partial_\beta^{\hat{}})=a^{*\alpha}l_{\beta\cdot}^{\cdot\gamma}g^*_{\alpha\gamma} \tag{19}$$

if frame ∂_α^* is orthogonal, we arrive at the following equations for the determination of $g^{\hat{}}_{\beta4}$:

$$\frac{\partial g^{\hat{}}_{\beta4}}{\partial\tau}=\sum_{\alpha=1}^{3}a^{*\alpha}l_{\beta\cdot}^{\cdot\alpha}$$

Relations (17) and (19) together with the initial data for $g^{\hat{}}_{\beta4}$ on the world line being considered completely determine the metric tensor components $g^{\hat{}}_{\beta4}$ and $g^{\hat{}}_{\alpha\beta}$ in the accompanying reference frame. From the preceding results it follows that the metric in the accompanying reference frame depends upon the geometric singularities of the world lines, since the four-dimensional acceleration vector **a** and, respectively, the three-dimensional acceleration vector **a*** measured by the accelerometer are determined by the four-dimensional curvature of world line C.

The problem of determining the law of motion of the moving medium from the viewpoint of the given observer is a navigation problem which can be solved if we accept that the components of the metric tensors and of the matrix $l_{\alpha\cdot}^{\cdot\beta}$ are known. Using these data we can also calculate all the components, introduced initially in the observer system or in the accompanying moving system, of the tensor characteristics of the physical phenomena in each of these reference frames. In 1943 Tkachev [4] (*) first posed and solved the general navigation problem "without systematic errors" (**) in the framework of Newtonian mechanics for the arbitrary motions of a rigid body.

To solve the general navigation problem within the frameworks of the STR and the GTR we rely on Eq. (18) (see [3]). Hoag and Wrigley [5] examined the navigation problem in the special case of the rectilinear motion of a material point. At each point of the world line being examined the unit orthogonal bases ∂_i in the Fermi coordinate system and the local inertial orthogonal unit vectors of bases ∂_i^* are interrelated by the Lorentz transformation. Since the spatial vectors of bases ∂_α and ∂_α^* preserve their orientation in three-dimensional space when passing from one point of the world line to another, we can take it that for both the three-dimensional vector frames in the three-dimensional space their orientations are the same at all points of the world line. (Their orientations vary in the four-dimensional space).

The bases ∂_i and ∂_i^* can be different only at the expense of the translation motion of the three-dimensional frame ∂_α^* relative to frame ∂_i. By $\mathbf{v} = v^\alpha \partial_\alpha$ we denote the vector of this translation motion. The corresponding transformation of the vectors of the basis and of the matrix of this Lorentz transformation for finite v^α have the form [6]

$$\partial_i^* = d_i^{\cdot j}\partial_j, \quad \| d_{i\cdot}^{\cdot j} \| = \left\| \begin{matrix} 1 + kv^1v^1 & kv^1v^2 & kv^1v^3 & \frac{v^1}{\sqrt{c^2 - v^2}} \\ kv^2v^1 & 1 + kv^2v^2 & kv^2v^3 & \frac{v^2}{\sqrt{c^2 - v^2}} \\ kv^3v^1 & kv^3v^2 & 1 + kv^3v^3 & \frac{v^3}{\sqrt{c^2 - v^2}} \\ \frac{v^1}{\sqrt{c^2 - v^2}} & \frac{v^2}{\sqrt{c^2 - v^2}} & \frac{v^3}{\sqrt{c^2 - v^2}} & \frac{c}{\sqrt{c^2 - v^2}} \end{matrix} \right\| \tag{20}$$

$$v^2 = (v^1)^2 + (v^2)^2 + (v^3)^2, \quad k = \frac{1}{v^2}\left[\frac{c}{\sqrt{c^2 - v^2}} - 1\right]$$

On the basis of (18) and (20) we obtain the equations for the components v^α at different points of world line C

$$v^\alpha = \frac{dy^\alpha}{dt}, \quad dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}}$$

$$\frac{d^2y^\alpha}{d\tau^2} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{d}{dt}\frac{v^\alpha}{\sqrt{1 - v^2/c^2}} = d_{\beta\cdot}^{\cdot\alpha} a^{*\beta} \tag{21}$$

*) Tkachev, L. I., On the theory of spatial orientation in instrument flight with the aid of pendulum-gyroscope systems. Master's Thesis, Bauman MVTU, 1944.

**) "Without systematic errors" signifies a regular recalculation for the observer of the measurement data obtained in tests in the accompanying reference frame by ideal instruments, i.e., instruments acting without instrument errors. Algorithms for solving individual navigation problems, proposed before Tkachev's work, contained systematic errors when applied to the general case.

Here $d\tau$ is an increment of proper time at point M on the world line, dt is an increment of proper time in the reference frame y^i with basis ϑ_i. It is easy to verify that the complementary equation (21), corresponding to $i = 4$, is satisfied identically by virtue of Eqs. (21) and of the connection $d\tau = \sqrt{1 - v^2 / c^2}\, dt$. Without loss of generality, when integrating the system of Eqs. (21) we can assume that we have $v_0{}^\alpha = 0$ for some typical instant τ_0. This can always be achieved by choosing a constant orientation of basis ϑ_i and, respectively, of vector ϑ_4 in the Fermi coordinate system, if we set $\vartheta_4 = \mathbf{u}\,(\tau_0)$.

After integrating Eqs. (21) for the given world line $\xi^\alpha = \text{const}$ we find

$$y^i = f^i\,(t) \tag{22}$$

The law of motion (22) gives a solution of the navigation problem within the framework of the STR if we accept that the coordinate system y^i is a global cartesian reference frame. To obtain the solution of the navigation problem in the GTR it is necessary to further transform the coordinates $x^i\,(y^j)$. Since

$$\vartheta_j = \frac{\partial \mathbf{r}}{\partial y^j} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial y^j} = \vartheta_i^\circ b_{j.}^{\cdot i} \tag{23}$$

from equalities (18) follows

$$\frac{d^2x^i}{ds^2} + \frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma^i_{jk}\,(x^l) = b_{j.}^{\cdot i}\frac{d^2y^j}{ds^2} = \frac{1}{c^2} b_{j.}^{\cdot i} d^j_{\alpha .} a^{*\alpha} \tag{24}$$

We obtain the equations for determining $b_{j.}^{\cdot i}$ from (23) with due regard to the constancy of vectors ϑ_j along C.

Let $\omega^1, \omega^2, \omega^3, \omega^4$ be the components of some arbitrary constant vector $\boldsymbol{\omega}$ in the constant basis ϑ_i. From (23) we can write: $\boldsymbol{\omega} = \vartheta_i^\circ b_j{}^{\cdot i}\ \omega^j$. Since

$$0 = \frac{d\boldsymbol{\omega}}{ds} = \frac{d b_{j.}^{\cdot i}\omega^j}{ds}\vartheta_i^\circ + b_{j.}^{\cdot k}\omega^j\Gamma^i_{kl}\frac{dx^l}{ds}\vartheta_i^\circ$$

along C, hence we obtain

$$\frac{db_{j.}^{\cdot i}}{ds} + b_{j.}^{\cdot k}\Gamma^i_{kl}\frac{dx^l}{ds} = 0 \tag{25}$$

Thus, for $v^\alpha\,(s)$, $x^i\,(s)$ and $b_{j.}{}^{\cdot i}\,(s)$ in the GTR we obtain a system of ordinary differential equations (21), (24) and (25) of order $3 + 8 + 16 = 27$. The corresponding initial conditions for $b_{j\cdot 0}{}^i$ and for $x_0{}^i$, $\dot{x}_0{}^i$ and v_0^i are easily found.

The algorithm proposed above can be replaced by another, depending on the different make up of the given measurements obtained by the inertial devices in tests or by the theory developed in the accompanying reference frame. From the preceding theory it is also clear that with the aid of the components of the metric tensors $g_{ij}\,(x^k)$ and $g_{\hat{i}\hat{j}}\,(\xi^\alpha, \tau)$, of the acceleration vector $\mathbf{a}^* = \mathbf{a}\cdot c^2$ and of the matrices $l_{\alpha .}{}^{\beta}$, $d_{i.}{}^{\cdot j}$ and $b_{j.}{}^{\cdot i}$ introduced above we can compute the components of any tensor given in one of the bases ϑ_i, $\bar{\vartheta}_i$, $\vartheta_i{}^*$, ϑ_i and $\vartheta_i{}^\circ$ on any other basis.

In the Fermi coordinates in the Riemannian space we can shift the vectors and tensors in bases ϑ_i along the fundamental world line, preserving the unchangeability of their components, as well as in the Euclidean space when using cartesian systems.

To pass from the accompanying reference frame to the local proper reference frame (or vice versa) we need to know only **a*** and the matrix $l_{\alpha}{}^{\cdot\beta}$. The determination of these data or of quantities equivalent to them is possible by mechanical means with the use of inertial or other devices or is possible theoretically on the basis of solving the appropriate problems by means of dynamic equations within the framework of physical models in the natural setting of the problems, implying that the unessential participation of the properties and states of the constant observer are excluded in the original laws. The accompanying reference frame, their three-dimensional metric and the proper time are connected with the physical relations and with the direct perceptions of the direct participants of the events from the physical essence of the processes and phenomena in the material bodies. These relations and perceptions must be contrasted with the random interpretations depending on the arbitrariness in the choice of the moving observers and their experiences, on their three-dimensional metric and on the flow of their proper time.

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